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The Gervais–Neveu–Felder equation for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra

A Chakrabarti† and R Chakrabarti‡

† Centre de Physique Théorique§, Ecole Polytechnique, 91128 Palaiseau Cédex, France
‡ Department of Theoretical Physics, University of Madras, Guindy Campus, Madras 600 025, India

E-mail: chakra@cpht.polytechnique.fr

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Abstract. Using a contraction procedure, we construct a twist operator that satisfies a shifted cocycle condition, and leads to the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra. The corresponding universal $\mathcal{R}_h(y)$ matrix obeys a Gervais–Neveu–Felder equation associated with the $U_{h;y}(sl(2))$ algebra. For a class of representations, the dynamical Yang–Baxter equation may be expressed as a compatibility condition for the algebra of the Lax operators.

Recently a class of invertible maps between the classical $sl(2)$ and the non-standard Jordanian $U_h(sl(2))$ algebras has been obtained [1–3]. The classical and the Jordanian coalgebraic structures may be related [2–5] by the twist operators corresponding to these maps. Following the first twist leading from the classical to the Jordanian Hopf structure, it is possible to envisage a second twist leading to a quasi-Hopf quantization of the Jordanian $U_h(sl(2))$ algebra. By explicitly constructing the appropriate universal twist operator that satisfies a shifted cocycle condition, we here obtain the Gervais–Neveu–Felder (GNF) equation satisfied by the universal $\mathcal{R}$ matrix of a one-parametric quasi-Hopf deformation of the $U_h(sl(2))$ algebra.

The GNF equation corresponding to the standard Drinfeld–Jimbo deformed $U_q(sl(2))$ algebra was studied in the context of Liouville field theory [6], quantization of the Kniznik–Zamolodchikov–Bernard equation [7] and quantization of the Calogero–Moser model in the R matrix formalism [8]. The general construction of the twist operators leading to the GNF equation corresponding to the quasi-triangular standard Drinfeld–Jimbo deformed $U_q(\mathfrak{g})$ algebras and superalgebras was obtained in [9–12].

For the sake of completeness, we start by enlisting the general properties of a quasi-Hopf algebra $\mathcal{A}$ [13]. For all $a \in \mathcal{A}$ there exists an invertible element $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ and the elements $(\alpha, \beta) \in \mathcal{A}$, such that

$$\begin{aligned}
&(\mathrm{id} \otimes \triangle) \triangle (a) = \Phi(\triangle \otimes \mathrm{id})(\triangle(a))\Phi^{-1} \\
&(\mathrm{id} \otimes \mathrm{id} \otimes \triangle)(\Phi)(\triangle \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (1 \otimes \Phi)(\mathrm{id} \otimes \triangle \otimes \mathrm{id})(\Phi)(\Phi \otimes 1) \\
&(\varepsilon \otimes \mathrm{id}) \circ \triangle = \mathrm{id} \\
&(\mathrm{id} \otimes \varepsilon) \circ \triangle = \mathrm{id} \\
&\sum_r S(a_r^{(1)})\alpha a_r^{(2)} = \varepsilon(a)\alpha
\end{aligned} \qquad (1)$$

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$$\sum_r a_r^{(1)} \beta S(a_r^{(2)}) = \varepsilon(a)\beta$$
$$\sum_r X_r^{(1)} \beta S(X_r^{(2)}) \alpha X_r^{(3)} = 1$$
$$\sum_r S(\bar{X}_r^{(1)}) \alpha \bar{X}_r^{(2)} \beta S(\bar{X}_r^{(3)}) = 1$$

where

$$\begin{aligned} &\triangle(a) = \sum_r a_r^{(1)} \otimes a_r^{(2)} \qquad \Phi = \sum_r X_r^{(1)} \otimes X_r^{(2)} \otimes X_r^{(3)} \\ &\Phi^{-1} = \sum_r \bar{X}_r^{(1)} \otimes \bar{X}_r^{(2)} \otimes \bar{X}_r^{(3)}. \end{aligned} \tag{2}$$

A quasi-triangular quasi-Hopf algebra is equipped with a universal $\mathcal{R}$ matrix satisfying

$$\begin{aligned} &\triangle^{op}(a) = \mathcal{R} \triangle (a) \mathcal{R}^{-1} \\ &(\mathrm{id} \otimes \triangle)(\mathcal{R}) = \Phi_{231}^{-1} \mathcal{R}_{13} \Phi_{213} \mathcal{R}_{12} \Phi_{123}^{-1} \\ &(\triangle \otimes \mathrm{id})(\mathcal{R}) = \Phi_{312} \mathcal{R}_{13} \Phi_{132}^{-1} \mathcal{R}_{23} \Phi_{123}. \end{aligned} \tag{3}$$

The algebra is known as triangular if the additional relation

$$\mathcal{R}_{21} = \mathcal{R}^{-1} \tag{4}$$

is satisfied. In a quasi-triangular quasi-Hopf algebra, the universal $\mathcal{R}$ matrix satisfies the quasi-Yang–Baxter equation

$$\mathcal{R}_{12} \Phi_{312} \mathcal{R}_{13} \Phi_{132}^{-1} \mathcal{R}_{23} \Phi_{123} = \Phi_{321} \mathcal{R}_{23} \Phi_{231}^{-1} \mathcal{R}_{13} \Phi_{213} \mathcal{R}_{12}. \tag{5}$$

An invertible twist operator $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfying the relation

$$(\varepsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{F}) \tag{6}$$

performs a gauge transformation as follows:

$$\begin{aligned} &\triangle_{\mathcal{F}}(a) = \mathcal{F} \triangle (a) \mathcal{F}^{-1} \\ &\Phi_{\mathcal{F}} = \mathcal{F}_{23} (\mathrm{id} \otimes \triangle)(\mathcal{F}) \Phi (\triangle \otimes \mathrm{id})(\mathcal{F}^{-1}) \mathcal{F}_{12}^{-1} \\ &\alpha_{\mathcal{F}} = \sum_r S(\bar{f}_r^{(1)}) \alpha \bar{f}_r^{(2)} \\ &\beta_{\mathcal{F}} = \sum_r f_r^{(1)} \beta S(f_r^{(2)}) \\ &\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1} \end{aligned} \tag{7}$$

where

$$\mathcal{F} = \sum_r f_r^{(1)} \otimes f_r^{(2)} \qquad \mathcal{F}^{-1} = \sum_r \bar{f}_r^{(1)} \otimes \bar{f}_r^{(2)}. \tag{8}$$

The Jordanian Hopf algebra $U_h(sl(2))$ is generated by the elements $(T^{\pm 1} (= \mathrm{e}^{\pm hX}), Y, H)$, satisfying the algebraic relations [14]

$$[H, T^{\pm 1}] = T^{\pm 2} - 1 \qquad [H, Y] = -\tfrac{1}{2}(Y(T + T^{-1}) + (T + T^{-1})Y) \qquad [X, Y] = H \tag{9}$$

whereas the coalgebraic properties are given by [14]

$$\begin{aligned} &\triangle(T^{\pm 1}) = T^{\pm 1} \otimes T^{\pm 1} \qquad \triangle (Y) = Y \otimes T + T^{-1} \otimes Y \\ &\triangle(H) = H \otimes T + T^{-1} \otimes H \\ &\varepsilon(T^{\pm 1}) = 1 \qquad \varepsilon(Y) = \varepsilon(H) = 0 \\ &S(T^{\pm 1}) = T^{\mp 1} \qquad S(Y) = -TYT^{-1} \qquad S(H) = -THT^{-1}. \end{aligned} \tag{10}$$

The universal $\mathcal{R}_h$ matrix of the triangular Hopf algebra $U_h(sl(2))$ is given in a convenient form [15] by

$$\mathcal{R}_h = \exp(-hX \otimes TH)\exp(hTH \otimes X). \tag{11}$$

An invertible nonlinear map of the generating elements of the $U_h(sl(2))$ algebra on the elements of the classical $U(sl(2))$ algebra plays a pivotal role in the present work. The map reads [2]

$$T = \tilde{T} \qquad Y = J_- - \tfrac{1}{4}h^2 J_+(J_0^2 - 1) \qquad H = (1 + (hJ_+)^2)^{1/2} J_0 \tag{12}$$

where $\tilde{T} = hJ_+ + (1 + (hJ_+)^2)^{1/2}$. The elements $(J_\pm, J_0)$ are the generators of the classical $sl(2)$ algebra

$$[J_0, J_\pm] = \pm 2J_\pm \qquad [J_+, J_-] = J_0. \tag{13}$$

The twist operator specific to the map (12), transforming the trivial classical $U(sl(2))$ coproduct structure to the non-cocommuting coproduct properties (10) of the Jordanian $U_h(sl(2))$ algebra, has been obtained [3,4] as a series expansion in powers of h. The transforming operator between the two above-mentioned antipode maps has been obtained [4] in a closed form.

Our present derivation of the GNF equation corresponding to the Jordanian $U_h(sl(2))$ algebra closely parallels the description in [8]. These authors obtained the solutions of the GNF equation in the case of the standard Drinfeld–Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra by constructing the universal twist operator depending on a parameter x:

$$\mathcal{F}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{(q - q^{-1})^k}{[k]_q!} x^{2k} q^{k(k+1)/2} \left[\prod_{l=1}^{k}(1 \otimes 1 - x^2 q^{2l} 1 \otimes q^{2\mathcal{J}_0})^{-1}\right] \times q^{\frac{k}{2}\mathcal{J}_0}\mathcal{J}_+^k \otimes q^{\frac{3k}{2}\mathcal{J}_0}\mathcal{J}_-^k \tag{14}$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. The generators of the $U_q(sl(2))$ algebra satisfy [13] the relations

$$q^{\mathcal{J}_0}\mathcal{J}_\pm q^{-\mathcal{J}_0} = q^{\pm 2}\mathcal{J}_\pm \qquad [\mathcal{J}_+, \mathcal{J}_-] = [\mathcal{J}_0]_q. \tag{15}$$

A key ingredient in our method is the contraction technique developed in [2], where a matrix G

$$G = E_q(\eta\mathcal{J}_+) \otimes E_q(\eta\mathcal{J}_+) \qquad \eta = \frac{h}{q - 1} \tag{16}$$

performs a similarity transformation on the universal $\mathcal{R}_q$ matrix of the $U_q(sl(2))$ algebra [13]. The twisted exponential $E_q(\chi)$ reads

$$E_q(\chi) = \sum_{n=0}^{\infty} \frac{\chi^n}{[n]_q!}. \tag{17}$$

The transforming matrix G is singular in the $q \to 1$ limit. The transformed $R_h^{j_1;j_2}$ matrix for an arbitrary $(j_1; j_2)$ represention

$$R_h^{j_1;j_2} = \lim_{q\to 1}[G^{-1} R_q^{j_1;j_2} G] \tag{18}$$

is, however, nonsingular and coincides, on account of the map (12), with the result obtained directly from the expression (11) of the universal $\mathcal{R}_h$ matrix. In the above contraction process the following two identities play a crucial role:

$$\begin{aligned}(E(\eta\mathcal{J}_+))^{-1} q^{\alpha\mathcal{J}_0/2} E(\eta\mathcal{J}_+) &= \mathcal{T}_{(\alpha)} q^{\alpha\mathcal{J}_0/2}\\ (E(\eta\mathcal{J}_+))^{-1}\mathcal{J}_- E(\eta\mathcal{J}_+) &= -\frac{\eta}{q - q^{-1}}(\mathcal{T}_{(1)} q^{\mathcal{J}_0} - \mathcal{T}_{(-1)} q^{-\mathcal{J}_0}) + \mathcal{J}_-\end{aligned} \tag{19}$$

where $\mathcal{T}_{(\alpha)} = (E(\eta\mathcal{J}_+))^{-1}E(q^\alpha\eta\mathcal{J}_+)$. In the $q \to 1$ limit, it may be proved [2]

$$\lim_{q\to 1}\mathcal{T}_{(\alpha)} = \tilde{T}^\alpha = T^\alpha. \tag{20}$$

The second equality in (20) follows from the map (12).

Using the contraction scheme discussed above we now obtain a one-parametric twist operator $\mathcal{F}_h(y) \in U_h(sl(2)) \otimes U_h(sl(2))$, which satisfies a shifted cocycle condition. The twist operator $\mathcal{F}_h(y)$ gauge transforms *à la* (7) the Jordanian Hopf algebra $U_h(sl(2))$ to a quasi-Hopf $U_{h;y}(sl(2))$ algebra and the transformed universal $\mathcal{R}_h(y)$ matrix satisfies the corresponding GNF equation. To this end we first compute

$$\tilde{\mathcal{F}}(y) = \lim_{q\to 1}(G^{-1}\mathcal{F}(x)G)_{x^2=y(q-1)} \tag{21}$$

where $\mathcal{F}(x)$ is given by (14). A new feature here is the reparametrization described by

$$y = \frac{x^2}{q-1} \tag{22}$$

which is necessary for obtaining a *nonsingular* result in the $q \to 1$ limit. In (22) we assume that $x \to 0$ in the $q \to 1$ limit in such a way that y remains finite. Following the above procedure in the said limit we obtain

$$\tilde{\mathcal{F}}(y) = \sum_{k=0}^{\infty}\frac{(hy)^k}{k!}(\tilde{T}J_+)^k \otimes (\tilde{T}^3(\tilde{T} - \tilde{T}^{-1}))^k. \tag{23}$$

The rhs of (23) is interpreted on account of the map (12) as an element of $U_h(sl(2)) \otimes U_h(sl(2))$. Identifying this in the above sense with the twist operator $\mathcal{F}_h(y)(=\tilde{\mathcal{F}}(y))$ we now obtain the crucial result

$$\mathcal{F}_h(y) = \exp\left(\frac{y}{2}(1 - T^2) \otimes (T^2 - T^4)\right). \tag{24}$$

The above twist operator $\mathcal{F}_h(y)$ satisfies the property (6). Following the arguments in [8] we express $\mathcal{F}_h(y)$ as a shifted coboundary

$$\mathcal{F}_h(y) = \triangle(\mathcal{M}(y))(1 \otimes \mathcal{M}^{-1}(y))(\mathcal{M}^{-1}(yT^4_{(2)}) \otimes 1) \tag{25}$$

where the expression for the boundary reads

$$\mathcal{M}(y) = \exp\left(\frac{y}{2}(1 - T^2)\right). \tag{26}$$

The operator $\mathcal{F}_h(y)$ given by (24) satisfies the following shifted cocycle condition:

$$(1 \otimes \mathcal{F}_h(y))[(\mathrm{id} \otimes \triangle)\mathcal{F}_h(y)] = (\mathcal{F}_h(yT^4_{(3)}) \otimes 1)[(\triangle \otimes \mathrm{id})\mathcal{F}_h(y)]. \tag{27}$$

Following (7) the transformed coproduct property may now be read as

$$\triangle_y(a) = \mathcal{F}_h(y) \triangle (a)\mathcal{F}_h^{-1}(y) \qquad \text{for all} \quad a \in U_{h;y}(sl(2)). \tag{28}$$

It may now be shown that the shifted cocycle condition is a consequence of the following shifted coassociativity property:

$$(\mathrm{id} \otimes \triangle_y) \circ \triangle_y(a) = (\triangle_{yT^4_{(3)}} \otimes \mathrm{id}) \circ \triangle_y(a). \tag{29}$$

Following (7) the gauge-transformed universal $\mathcal{R}_h(y)$ matrix for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra reads

$$\mathcal{R}_h(y) = \mathcal{F}_{h21}(y)\mathcal{R}_h\mathcal{F}_h^{-1}(y). \tag{30}$$

The coassociator $\Phi(y)$ corresponding to the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra may be obtained for the above construction of the twist operator obeying the shifted cocycle condition (27). Using (7), (24) and (27) we obtain

$$\begin{aligned}\Phi(y) &= \mathcal{F}_{h12}(yT^4_{(3)})\mathcal{F}^{-1}_{h12}(y)\\ &= \exp\left[-\frac{y}{2}(1-T^2)\otimes(T^2-T^4)\otimes(1-T^4)\right].\end{aligned} \tag{31}$$

The elements $\alpha(y)$ and $\beta(y)$, characterizing the antipode map of the $U_{h;y}(sl(2))$ algebra, may be similarly obtained from (7), (10) and (24):

$$\alpha(y) = \exp\left[\frac{y}{2}(1-T^2)^2\right] \qquad \beta(y) = \exp\left[-\frac{y}{2}(1-T^{-2})^2\right]. \tag{32}$$

Using the gauge transformation property of the universal $\mathcal{R}$ matrix in (7) and our construction (24) of the twist operator, we now discuss the GNF equation associated with the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra. The relations (7), (24) and (31) lead to the transformation property

$$\mathcal{R}_{h12}(yT^4_{(3)}) = \Phi_{213}(y)\mathcal{R}_{h12}(y)\Phi^{-1}_{123}(y). \tag{33}$$

Now the quasitriangularity property of the $U_{h;y}(sl(2))$ algebra implies via (3), (31) and (33) the following relations:

$$\begin{aligned}(\mathrm{id}\otimes\triangle_y)\mathcal{R}_h(y) &= \mathcal{F}_{h23}(y)\mathcal{F}^{-1}_{h23}(yT^4_{(1)})\mathcal{R}_{h13}(y)\mathcal{R}_{h12}(yT^4_{(3)})\\ (\triangle_y\otimes\mathrm{id})\mathcal{R}_h(y) &= \mathcal{R}_{h13}(yT^4_{(2)})\mathcal{R}_{h23}(y)\mathcal{F}_{h12}(yT^4_{(3)})\mathcal{F}^{-1}_{h12}(y).\end{aligned} \tag{34}$$

Using the transformation property (33) we may now recast the quasi-Yang–Baxter equation (5) as the GNF equation associated with the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra:

$$\mathcal{R}_{h12}(y)\mathcal{R}_{h13}(yT^4_{(2)})\mathcal{R}_{h23}(y) = \mathcal{R}_{h23}(yT^4_{(1)})\mathcal{R}_{h13}(y)\mathcal{R}_{h12}(yT^4_{(3)}). \tag{35}$$

We now briefly consider the solutions of the above GNF equation (35). Using the universal $\mathcal{R}_h(y)$ matrix (30), the twist operator $\mathcal{F}_h(y)$ in (24) and the map (12) of the generators of the $U_h(sl(2))$ algebra on the corresponding classical elements, we may construct solutions of the GNF equation (35). As illustrations we describe the representations $R_h(y)$ for the $\frac{1}{2}\otimes j$ and the $1\otimes j$ cases. A $(2j+1)$-dimensional representation of the classical $sl(2)$ algebra (13)

$$\begin{aligned}&J_+|jm\rangle = (j-m)(j+m+1)|jm+1\rangle \qquad J_-|jm\rangle = |jm-1\rangle\\ &J_0|jm\rangle = m|jm\rangle\end{aligned} \tag{36}$$

now, via the map (12), immediately furnishes the corresponding $(2j+1)$-dimensional representation of the $U_h(sl(2))$ algebra (9). For the $j=\frac{1}{2}$ case, the generators remain undeformed. For the $j=1$ case, we list the representation of $U_h(sl(2))$ below.

$$\begin{aligned}&(j=1)\\ &X = \begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix} \qquad Y = \begin{pmatrix}0&\frac{1}{2}h^2&0\\1&0&-\frac{3}{2}h^2\\0&1&0\end{pmatrix}\\ &H = \begin{pmatrix}2&0&-4h^2\\0&0&0\\0&0&-2\end{pmatrix}.\end{aligned} \tag{37}$$

Using the above representations in the expression (30) of the universal $\mathcal{R}_h(y)$ matrix, we obtain

$$R_h^{\frac{1}{2};j}(y) = \begin{pmatrix}T & -hH+\frac{1}{2}h(T-T^{-1})(1+2y(1-T^4))\\ 0 & T^{-1}\end{pmatrix} \tag{38}$$

and

$$R_h^{1;j}(y) = \begin{pmatrix} T^2 & A & B \\ 0 & 1 & C \\ 0 & 0 & T^{-2} \end{pmatrix} \tag{39}$$

where

$$\begin{aligned} A &= -2hTH - 2hy(1-T^2)(1-T^4) \\ B &= -2h^2[T^2 - T^{-2} - 2TH(1-T^{-2}) - (TH)^2T^{-2}] \\ &\quad -4h^2y(1-T^2)(1+4T^{-2}-T^4) \\ &\quad -4h^2yTH(1-T^2)(T^2-T^{-2}) + 2h^2y^2(T-T^{-1})^2(1-T^4)^2 \\ C &= -2h(1-T^{-2}+THT^{-2}) + 2hy(1-T^2)(T^2-T^{-2}). \end{aligned} \tag{40}$$

From (38) it follows that the $R_h^{\frac{1}{2};\frac{1}{2}}$ matrix for the fundamental $(\frac{1}{2};\frac{1}{2})$ case does not depend on the parameter y. The $R_h(y)$ matrices for the higher representations, however, nontrivially depend on y. The $R_h(y)$ matrices satisfy an 'exchange symmetry' between the two sectors of the tensor product spaces:

$$(R_h^{j_1;j_2}(y))_{km,ln} = (R_{-h}^{j_2;j_1}(y))_{mk,nl}. \tag{41}$$

In the remaining part of the present work we recast the Jordanian GNF equation (35) as a compatibility condition for the algebra of L operators. Using a new parametrization $y = \exp(z)$, we perform a translation

$$\mathcal{R}_{h12}(z) \to \mathcal{R}_{h12}(z - 2hX_{(3)}) \tag{42}$$

to express (35) in a symmetric form

$$\begin{aligned} &\mathcal{R}_{h12}(z-2hX_{(3)})\mathcal{R}_{h13}(z+2hX_{(2)})\mathcal{R}_{h23}(z-2hX_{(1)}) \\ &\quad = \mathcal{R}_{h23}(z+2hX_{(1)})\mathcal{R}_{h13}(z-2hX_{(2)})\mathcal{R}_{h12}(z+2hX_{(3)}). \end{aligned} \tag{43}$$

This is equivalent to the Jordanian GNF equation (35) for the class of representations $\varrho_{j_1;j_2}$ satisfying the property

$$\varrho_{j_1;j_2}([(X_{(k)} + X_{(l)})\partial_z, \mathcal{R}_{hkl}(z)]) = 0. \tag{44}$$

Adopting the procedure in [8] we here use the following construction of the Lax operator for the $U_{h;y}(sl(2))$ algebra:

$$L_{13}(z) = \exp[-2h(2X_{(1)} + X_{(3)})\partial_z]\mathcal{R}_{h13}(z)\exp[2hX_{(3)}\partial_z] \tag{45}$$

where the subscript 3 denotes the quantum space. For the representations satisfying (44), relation (43) may be expressed in a Lax matrix form

$$R_{h12}^{j_1;j_2}(z-2hX_{(3)})L_{13}(z)L_{23}(z) = L_{23}(z)L_{13}(z)R_{h12}^{j_1;j_2}(z+2hX_{(3)}). \tag{46}$$

As illustrations we note that the representations $R_h^{\frac{1}{2};1}(z)$, $R_h^{1;\frac{1}{2}}(z)$ and $R_h^{1;1}(z)$ obtained from (38) and (39) satisfy the requirement (44).

To summarize, here we have constructed the Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra by explicitly obtaining the relevant twist operator via a contraction method. In the contraction method used here we start with the standard Drinfeld–Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra and use a suitable similarity transformation followed by a $q \to 1$ limiting process. In contrast to our earlier works [2–4] relating to the contraction mechanism, a distinctive point here is that the reparametrization as obtained in (22) is essential for obtaining a *nonsingular* twist operator for the $U_{h;y}(sl(2))$ algebra in the $q \to 1$ limit. Our contraction method has an advantage in that it furnishes the dynamical quantities for the Jordanian quasi-Hopf $U_{h;y}(sl(2))$

algebra from the corresponding quantities of the standard Drinfeld–Jimbo deformed quasi-Hopf $U_{q;x}(sl(2))$ algebra. The present twist operator associated with the $U_{h;y}(sl(2))$ algebra satisfies a shifted cocycle condition. The universal $\mathcal{R}_h(y)$ matrix satisfies the GNF equation associated with the $U_{h;y}(sl(2))$ algebra. For a special class of representations, the GNF equation may be recast as a compatibility condition of the L operators. As an extension of the present work, a similar formalism may be developed to describe a quasi-Hopf quantization of the coloured Jordanian deformed $gl(2)$ algebra considered in [4,16,17]. A similar construction of the twist operator associated with the quasi-Hopf deformation of an arbitrary Jordanian $sl_h(N)$ algebra may also be attempted following the discussion in [2].

Lastly we comment on the possible applications of the quasi-Hopf $U_{h;y}(sl(2))$ algebra discussed here. Using the representations of the coalgebra and the Casimir operator for the Jordanian deformation of the $sl(2)$ algebra, a nonstandard integrable deformation of the XXX hyperbolic Gaudin system has been recently obtained [18]. The Jordanian quasi-Hopf $U_{h;y}(sl(2))$ algebra obtained here may be similarly used to obtain a new one-parametric family of exactly integrable Hamiltonians using the transformed coalgebraic structure (28). Finally, the dual of a quasi-Hopf algebra is evidently something that is associative only up to conjugation in a suitable convolution algebra by a 3-cocycle Φ. Our work on the quasi-Hopf $U_{h;y}(sl(2))$ algebra may lead to a nonassociative generalization of the noncommutative differential geometry of the h-deformed quantum space studied in [19].

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